

# Higher order mean curvature estimates for bounded complete hypersurfaces \*

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## Abstract

We obtain sharp estimates involving the mean curvatures of higher order of a complete bounded hypersurface immersed in a complete Riemannian manifold. Similar results are also given for complete spacelike hypersurfaces in Lorentzian ambient spaces.

Estimates for the  $k$ -mean curvatures  $H_k$  of higher order of a compact hypersurface in a complete Riemannian manifold have been subsequently obtained by Vlachos [14], Veeravalli [13], Fontenele-Silva [9], Roth [12] and Ranjbar-Motlagh [11]. In this paper, we generalize a result given in the latter that we describe next.

Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be a codimension one isometric immersion between complete Riemannian manifolds. Assume that the hypersurface lies inside a closed geodesic ball  $B_{\bar{M}}(r)$  of radius  $r$  and center  $o \in \bar{M}^{n+1}$  and that  $0 < r < \min\{\text{inj}_{\bar{M}}(o), \pi/2\sqrt{b}\}$  where  $\text{inj}_{\bar{M}}(o)$  is the injectivity radius at  $o$  and  $\pi/2\sqrt{b}$  is replaced by  $+\infty$  if  $b \leq 0$ . Suppose also that there is a point  $p_0 \in M^n$  such that  $f(p_0) \in S_{\bar{M}}(r)$  where  $S_{\bar{M}}(r)$  is the boundary of  $B_{\bar{M}}(r)$ . In the context of this paper, this is a slightly weaker assumption than asking  $M^n$  to be compact. Let  $K_{\bar{M}}^{\text{rad}}$  denote the radial sectional curvatures in  $B_{\bar{M}}(r)$  along geodesics issuing from the center and assume that  $K_{\bar{M}}^{\text{rad}} \leq b$  for some constant

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$b \in \mathbb{R}$ . Assume also that  $H_{k+1} \neq 0$  everywhere for some  $2 \leq k \leq n-1$ . In this situation, it turns that the  $p_0$  is an elliptic point. This means that the second fundamental form of  $f$  at  $p_0$  with respect to the inner pointing orientation is positive definite. From the well-known Garding inequalities it follows that  $H_j > 0$  for  $1 \leq j \leq k+1$ .

In the above situation, it was shown in Theorem 4.2 in [11] that

$$\sup_M \left( \frac{H_{j+1}}{H_j} \right) \geq C_b(r)$$

for any  $1 \leq j \leq k$ , where the constant  $C_b(r)$  given by (24) below is the mean curvature of a geodesic sphere of radius  $r$  in a simply connected space form of sectional curvature  $b$ . Moreover, if equality holds for some  $j$  then it follows that  $M^n = S_{\bar{M}}(r)$ .

Our main goal in this paper is to replace the assumption of compactness of the submanifold by the much weaker of completeness. The tool that makes this generalization possible is an Omori-Yau type maximum principle for trace type differential operators in the spirit of those due to Albanese, Alías and Rigoli [1] (see Theorem 3 below).

The following is a consequence of the quite more general result given in Section 2. Here, the more general but technical assumptions made in Theorem 5 of Section 2 take a simpler geometric form.

**Theorem 1.** *Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be an isometric immersion between complete Riemannian manifolds such that  $f(M) \subset B_{\bar{M}}(r)$ . Assume  $H_{k+1} \neq 0$  everywhere for some  $2 \leq k \leq n-1$  and that the sectional curvatures satisfy  $K_M \geq K > -\infty$  and  $K_M^{\text{rad}} \leq b$  for some constant  $b \in \mathbb{R}$ . If  $f$  has an elliptic point, then*

$$\sup_M {}^{j+1}\sqrt{H_{j+1}} \geq \sup_M \left( \frac{H_{j+1}}{H_j} \right) \geq C_b(r), \quad 1 \leq j \leq k. \quad (1)$$

Moreover, if there exists a point  $p_0 \in M^n$  such that  $f(p_0) \in S_{\bar{M}}(r)$  and  $\sup_M (H_{j+1}/H_j) = C_b(r)$  for some  $j$  then  $M^n = S_{\bar{M}}(r)$ .

In the second part of the paper and motivated, among others, by the results in [2] and [3], we show that similar estimates than in the Riemannian case hold for complete spacelike hypersurfaces in Lorentzian ambient spaces.

# 1 A maximum principle

The aim of this section is to introduce the main analytic ingredient for the proof of our results. It consists in a maximum principle of Omori-Yau type in the spirit of those given in [1] that applies to trace type operators like those described in the sequel.

Let  $M^n$  be a Riemannian manifold and  $\nabla$  the Levi-Civita connection. For  $u \in C^2(M)$  let  $\text{hess } u: TM \rightarrow TM$  be the symmetric operator given by  $\text{hess } u(X) = \nabla_X \nabla u$  and by  $\text{Hess } u: TM \times TM \rightarrow C^0(M)$  the metrically equivalent bilinear form given by

$$\text{Hess } u(X, Y) = \langle \text{hess } u(X), Y \rangle.$$

Associated to a symmetric tensor  $P: TM \rightarrow TM$ , we consider the second order differential operator  $L: C^2(M) \rightarrow C^0(M)$  given by  $L = \text{Tr}(P \circ \text{hess})$ . Observe that  $L(u) = \text{div}(P \nabla u) - \langle \text{div } P, \nabla u \rangle$ , where  $\text{div } P = \text{Tr} \nabla P$ . This implies that  $L$  is (semi-)elliptic if and only if  $P$  is positive (semi-)definite. The following result is Theorem B together with Remark 1.2 in [1].

**Theorem 2.** *Let  $M^n$  be a Riemannian manifold and let  $L = \text{Tr}(P \circ \text{hess})$  be a semi-elliptic linear operator. Let  $q \in C^0(M)$  be nonnegative such that  $q > 0$  outside a compact set. Assume that there exists  $\gamma \in C^2(M)$  with the following properties:*

- (a)  $\gamma(p) \rightarrow +\infty$  as  $p \rightarrow \infty$ ,
- (b)  $\|\nabla \gamma\| \leq G(\gamma)$  off a compact set,
- (c)  $qL\gamma \leq G(\gamma)$  off a compact set

where  $G$  is a smooth function on  $[0, +\infty)$  such that:

- (i)  $G(0) > 0$ , (ii)  $G'(t) \geq 0$  and (iii)  $1/G(t) \notin L^1(+\infty)$ .

Then, for any function  $u \in C^2(M)$  with  $u^* = \sup_M u < +\infty$  there exists a sequence  $\{p_j\}_{j \in \mathbb{N}}$  in  $M^n$  such that

- (i)  $u(p_j) > u^* - \frac{1}{j}$ , (ii)  $\|\nabla u(p_j)\| < \frac{1}{j}$  and (iii)  $q(p_j)Lu(p_j) < \frac{1}{j}$ .

Following the terminology in [1], we say that the *q-Omori-Yau maximum principle holds on  $M^n$  for  $L$*  as above whenever the conclusions of Theorem 2 hold.

Let  $M^n$  be a complete noncompact Riemannian manifold. Denote by  $r(x)$  the distance function to a fixed reference point  $o \in M^n$ . Then  $r(x)$  satisfies assumptions (a) and (b) of Theorem 2. Although  $r(x)$  is not  $C^2$  in  $o$  and its cut locus  $\text{cut}(o)$ , one could think of it as a natural candidate for  $\gamma$ , under appropriate curvature assumptions. The technical difficulty arising from this choice, and related to the lack of smoothness, forces us to introduce a reasoning in some way similar to approaching the problem via viscosity solutions in order to get the following result.

**Theorem 3.** *Let  $M^n$  be a complete, non-compact Riemannian manifold and let  $r(x)$  be the Riemannian distance function from a reference point  $o \in M^n$ . Assume that the sectional curvature of  $M^n$  satisfies*

$$K_M(x) \geq -G^2(r(x)) \quad (2)$$

with  $G \in \mathcal{C}^1([0, +\infty))$  satisfying

$$(i) \ G(0) > 0, \quad (ii) \ G'(t) \geq 0 \quad \text{and} \quad (iii) \ 1/G(t) \notin L^1(+\infty).$$

Then, the *q-Omori-Yau maximum principle holds on  $M^n$  for any semi-elliptic operator of the form  $L = \text{Tr}(P \circ \text{hess})$  with  $\text{tr}P > 0$  on  $M^n$  where  $q = 1/\text{Tr}P$ .*

*Proof:* Let  $D_o = M^n \setminus \text{cut}(o)$  be the domain of normal geodesic coordinates centered at  $o$ . On  $D_o$  we have from (2) and the general Hessian comparison theorem [10, Theorem 2.3] that

$$\text{Hess}(r) \leq \frac{g'(r)}{g(r)}(\langle \cdot, \cdot \rangle - dr \otimes dr), \quad (3)$$

where  $g(t)$  is the (positive on  $\mathbb{R}^+ = (0, +\infty)$ ) solution of the Cauchy problem

$$\begin{cases} g''(t) - G^2(t)g(t) = 0, \\ g(0) = 0, \quad g'(0) = 1. \end{cases} \quad (4)$$

Letting

$$\psi(t) = \frac{1}{G(0)}(e^{\int_0^t G(s)ds} - 1)$$

we have  $\psi(0) = 0$ ,  $\psi'(0) = 1$  and

$$\psi''(t) - G^2(t)\psi(t) = \frac{1}{G(0)} \left( G^2(t) + G'(t) e^{\int_0^t G(s)ds} \right) \geq 0,$$

that is,  $\psi$  is a subsolution of (4). By the Sturm comparison theorem

$$\frac{g'(t)}{g(t)} \leq \frac{\psi'(t)}{\psi(t)} = G(t) \frac{e^{\int_0^t G(s)ds}}{e^{\int_0^t G(s)ds} - 1}.$$

Thus, we have

$$Lr(x) \leq \operatorname{tr} P(x) \frac{\psi'(r(x))}{\psi(r(x))} = \operatorname{tr} P(x) G(r(x)) \frac{e^{\int_0^{r(x)} G(s)ds}}{e^{\int_0^{r(x)} G(s)ds} - 1}.$$

Since  $G > 0$ ,  $G' \geq 0$  and  $\operatorname{tr} P \geq 0$ , we obtain

$$Lr(x) \leq \operatorname{tr} P(x) G(r(x) + 1) \frac{e^{\int_0^{r(x)} G(s)ds}}{e^{\int_1^{r(x)} G(s)ds} - 1}, \quad r(x) \geq 2. \quad (5)$$

Define

$$\varphi(t) = \int_0^t \frac{ds}{G(s+1)} \quad (6)$$

so that

$$\varphi'(t) = \frac{1}{G(t+1)} \quad \text{and} \quad \varphi''(t) \leq 0.$$

Set  $\gamma(x) = \varphi(r(x))$  on  $M^n \setminus \bar{B}_2$  and note that

$$\gamma(x) \rightarrow +\infty \text{ as } x \rightarrow \infty \quad (7)$$

because  $\varphi(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$  since  $1/G \notin L^1(+\infty)$ .

Using the formula  $L\varphi(u) = \varphi'(u)Lu + \varphi''(u)\langle P\nabla u, \nabla u \rangle$  and that  $P$  is positive semi-definite, we obtain from (5) that

$$L\gamma(x) \leq \varphi'(r(x))Lr(x) = \frac{1}{G(r(x)+1)}Lr(x) \leq \operatorname{tr} P(x) \frac{e^{\int_0^{r(x)} G(s)ds}}{e^{\int_1^{r(x)} G(s)ds} - 1}.$$

Since  $G \notin L^1(+\infty)$  we have

$$\sup_{t \geq 2} \frac{e^{\int_0^t G(s)ds}}{e^{\int_1^t G(s)ds} - 1} = \Lambda < +\infty. \quad (8)$$

We deduce that  $L\gamma(x) \leq \text{tr}P(x)\Lambda$ , i.e.,

$$q(x)L\gamma(x) \leq \Lambda \text{ on } D_o \cap (M^n \setminus \bar{B}_2). \quad (9)$$

Let  $u \in \mathcal{C}^2(M)$  with  $u^* = \sup_M u < +\infty$ . For a fixed  $\eta > 0$ , consider

$$A_\eta = \{x \in M^n : u(x) > u^* - \eta\}$$

and

$$B_\eta = \{x \in A_\eta : \|\nabla u(x)\| < \eta\}.$$

Since  $M^n$  is complete, we have from the Ekeland quasi-minimum principle (cf. [6]) that  $B_\eta \neq \emptyset$ . All we have to show is that

$$\inf_{B_\eta} \{q(x)Lu(x)\} \leq 0 \quad (10)$$

since this is equivalent to the claim of the theorem. To prove (10) we reason by contradiction. In fact, assume that

$$q(x)Lu(x) \geq \sigma_0 > 0 \text{ on } B_\eta. \quad (11)$$

First observe that  $u^*$  cannot be attained at a point  $x_0 \in M^n$ , for otherwise  $x_0 \in B_\eta$  but, since  $P$  is positive semi-definite, then  $q(x_0)Lu(x_0) \leq 0$  thus contradicting (11). Set

$$\Omega_t = \{x \in M^n : \gamma(x) > t\}.$$

Then  $\Omega_t^c = M^n \setminus \Omega_t$  is closed and hence compact by (7). Define

$$u_t^* = \max_{x \in \Omega_t^c} u(x).$$

Since  $u^*$  is not attained in  $M^n$  and  $\{\Omega_t^c\}$  is a nested family exhausting  $M^n$ , there is a divergent sequence  $\{t_j\}_{j \in \mathbb{N}} \subset [0, +\infty)$  such that

$$u_{t_j}^* \rightarrow u^* \text{ as } j \rightarrow +\infty, \quad (12)$$

and  $T_1 > 0$  sufficiently large such that  $u_{T_1}^* > u^* - \eta/2$  and  $\Omega_{T_1} \subset M^n \setminus \bar{B}_2$ . In particular, (9) holds on  $D_o \cap \Omega_{T_1}$ . Choose  $\alpha$  such that  $u_{T_1}^* < \alpha < u^*$ . Because of (12) we can find  $j$  sufficiently large such that  $T_2 = t_j > T_1$  and  $u_{T_2}^* > \alpha$ . Then, we select  $\delta > 0$  small enough so that

$$\alpha + \delta < u_{T_2}^*. \quad (13)$$

For  $\sigma > 0$  define

$$\gamma_\sigma(x) = \alpha + \sigma(\gamma(x) - T_1).$$

Then, we have

$$\gamma_\sigma(x) = \alpha \quad \text{for } x \in \partial\Omega_{T_1}$$

and from (9) for  $\sigma$  sufficiently small that

$$q(x)L\gamma_\sigma(x) = \sigma q(x)L\gamma(x) \leq \sigma\Lambda < \sigma_0 \quad \text{on } D_o \cap \Omega_{T_1}. \quad (14)$$

On  $\Omega_{T_1} \setminus \Omega_{T_2}$ , we have

$$\alpha \leq \gamma_\sigma(x) \leq \alpha + \sigma(T_2 - T_1).$$

Thus, choosing  $\sigma > 0$  sufficiently small so that

$$\sigma(T_2 - T_1) < \delta, \quad (15)$$

we obtain

$$\alpha \leq \gamma_\sigma(x) < \alpha + \delta \quad \text{on } \Omega_{T_1} \setminus \Omega_{T_2}.$$

For  $x \in \partial\Omega_{T_1}$  we have that  $\gamma_\sigma(x) = \alpha > u_{T_1}^* \geq u(x)$ . Hence,

$$(u - \gamma_\sigma)(x) < 0 \quad \text{on } \partial\Omega_{T_1}. \quad (16)$$

Let  $\bar{x} \in \Omega_{T_1} \setminus \Omega_{T_2}$  be such that  $u(\bar{x}) = u_{T_2}^* > \alpha + \delta$ . Then (13) and (15) yield

$$(u - \gamma_\sigma)(\bar{x}) \geq u_{T_2}^* - \alpha - \sigma(T_2 - T_1) > u_{T_2}^* - \alpha - \delta > 0.$$

Moreover, we have from (7) and  $u^* < +\infty$  for  $T_3 > T_2$  sufficiently large that

$$(u - \gamma_\sigma)(x) < 0 \quad \text{on } \Omega_{T_3} \quad (17)$$

Therefore,

$$m = \sup_{x \in \bar{\Omega}_{T_1}} (u - \gamma_\sigma)(x) > 0$$

is, in fact, a maximum attained at a point  $z_0$  in the compact set  $\bar{\Omega}_{T_1} \setminus \Omega_{T_3}$ .

From (16) we know that  $\gamma(z_0) > T_1$ . Thus, we have

$$u(z_0) = \gamma_\sigma(z_0) + m > \gamma_\sigma(z_0) > \alpha > u_{T_1}^* > u^* - \frac{\eta}{2},$$

and hence  $z_0 \in A_\eta \cap \Omega_{T_1}$ . Next, we have to distinguish two cases, according to  $z_0 \in D_o$  or  $z_0 \notin D_o$ .

If  $z_0 \in D_o$ , since  $z_0$  is a maximum for  $u - \gamma_\sigma$ , we have  $\nabla(u - \gamma_\sigma)(z_0) = 0$ . Using this fact, we have that  $z_0 \in B_\eta$  since

$$\|\nabla u(z_0)\| = \|\nabla \gamma_\sigma(z_0)\| = \sigma \varphi'(r(z_0)) \|\nabla r(z_0)\| = \frac{\sigma}{G(r(z_0) + 1)} \leq \frac{\sigma}{G(1)} < \eta$$

up to choosing  $\sigma$  sufficiently small. Since  $P$  is positive semi-definite and  $z_0$  is a maximum for  $u - \gamma_\sigma$ , we have  $Lu(z_0) \leq L\gamma_\sigma(z_0)$ , and this jointly with (9) yields

$$0 < \sigma_0 \leq q(z_0)Lu(z_0) \leq q(z_0)L\gamma_\sigma(z_0) < \sigma_0,$$

which is a contradiction and concludes the proof for this case.

In the case  $z_0 \notin D_o$  we reason as follows. Fix  $0 < \varepsilon < 1$  sufficiently small so that for the minimizing geodesic  $\varsigma$  parametrized by arclength and joining  $o$  with  $z_0$ , the point  $o_\varepsilon = \varsigma(\varepsilon) \neq z_0$  and  $z_0 \notin \text{cut}(o_\varepsilon)$ . Hence, the function  $r_\varepsilon(x) = \text{dist}(o_\varepsilon, x)$  is  $\mathcal{C}^2$  in a neighborhood of  $z_0$ . By the triangle inequality

$$r(x) \leq r_\varepsilon(x) + \varepsilon, \tag{18}$$

equality holding at  $z_0$ . With  $\varphi$  defined in (6) set

$$\gamma^\varepsilon(x) = \varphi(r_\varepsilon(x) + \varepsilon).$$

Since  $\varphi$  is increasing

$$\gamma(x) = \varphi(r(x)) \leq \varphi(r_\varepsilon(x) + \varepsilon) = \gamma^\varepsilon(x). \tag{19}$$

and

$$\gamma(z_0) = \gamma^\varepsilon(z_0). \tag{20}$$

Next consider the function

$$\gamma_\sigma^\varepsilon(x) = \alpha + \sigma(\gamma^\varepsilon(x) - T_1) (\geq \gamma_\sigma(x)).$$

Because of (19) and (20) we have in a neighborhood of  $z_0$  that

$$u(x) - \gamma_\sigma^\varepsilon(x) \leq u(x) - \gamma_\sigma(x) \leq m$$

and

$$u(z_0) - \gamma_\sigma^\varepsilon(z_0) = u(z_0) - \gamma_\sigma(z_0) = m.$$

Hence  $z_0$  is also a local maximum for  $u(x) - \gamma_\sigma^\varepsilon(x)$ . Thus,

$$\nabla u(z_0) = \nabla \gamma_\sigma^\varepsilon(z_0) \tag{21}$$



and

$$Lu(z_0) \leq L\gamma_\sigma^\varepsilon(z_0). \quad (22)$$

From (21) we deduce

$$\begin{aligned} \|\nabla u(z_0)\| &= \sigma \|\nabla \gamma^\varepsilon(z_0)\| = \sigma \varphi'(r_\varepsilon(z_0) + \varepsilon) \|\nabla r_\varepsilon(z_0)\| \\ &= \frac{\sigma}{G(r_\varepsilon(z_0) + \varepsilon + 1)} \leq \frac{\sigma}{G(1)} < \eta. \end{aligned}$$

Since we already knew that  $z_0 \in A_\eta$ , we conclude that  $z_0 \in B_\eta$ . Now we analyze (22). Because of (2), (18) and  $G' \geq 0$  we have

$$K_M(x) \geq -G^2(r(x)) \geq -G^2(r_\varepsilon(x) + \varepsilon).$$

Set  $G_\varepsilon(t) = G(t + \varepsilon)$  and consider the Cauchy problem (4) with  $G_\varepsilon$  instead of  $G$ . Again by the Hessian comparison theorem, on  $D_{o_\varepsilon}$  we have

$$Lr_\varepsilon(x) \leq \operatorname{tr} P(x) \frac{\psi'_\varepsilon(r_\varepsilon(x))}{\psi_\varepsilon(r_\varepsilon(x))} \quad \text{where} \quad \psi_\varepsilon(t) = \frac{1}{G_\varepsilon(0)} \left( e^{\int_0^t G_\varepsilon(s) ds} - 1 \right).$$

Observing that  $z_0 \in D_{o_\varepsilon}$ , we obtain using (8) that

$$\begin{aligned} L\gamma^\varepsilon(z_0) &\leq \varphi'(r_\varepsilon(z_0) + \varepsilon) Lr_\varepsilon(z_0) = \frac{1}{G(r_\varepsilon(z_0) + \varepsilon + 1)} Lr_\varepsilon(z_0) \\ &= \frac{1}{G(r(z_0) + 1)} Lr_\varepsilon(z_0) \leq \frac{\operatorname{tr} P(z_0)}{G(r(z_0) + 1)} \frac{\psi'_\varepsilon(r_\varepsilon(z_0))}{\psi_\varepsilon(r_\varepsilon(z_0))} \\ &= \frac{\operatorname{tr} P(z_0)}{G(r(z_0) + 1)} G_\varepsilon(r_\varepsilon(z_0)) \frac{e^{\int_0^{r_\varepsilon(z_0)} G(s+\varepsilon) ds}}{e^{\int_0^{r_\varepsilon(z_0)} G(s+\varepsilon) ds} - 1} \\ &= \operatorname{tr} P(z_0) \frac{G(r_\varepsilon(z_0) + \varepsilon)}{G(r(z_0) + 1)} \frac{e^{\int_\varepsilon^{r_\varepsilon(z_0)+\varepsilon} G(s) ds}}{e^{\int_\varepsilon^{r_\varepsilon(z_0)+\varepsilon} G(s) ds} - 1} \\ &= \operatorname{tr} P(z_0) \frac{G(r(z_0))}{G(r(z_0) + 1)} \frac{e^{\int_\varepsilon^{r(z_0)} G(s) ds}}{e^{\int_\varepsilon^{r(z_0)} G(s) ds} - 1} \\ &\leq \operatorname{tr} P(z_0) \frac{e^{\int_0^{r(z_0)} G(s) ds}}{e^{\int_1^{r(z_0)} G(s) ds} - 1} \leq \operatorname{tr} P(z_0) \Lambda. \end{aligned}$$

Thus,

$$L\gamma_\sigma^\varepsilon(z_0) = \sigma L\gamma^\varepsilon(z_0) \leq \operatorname{tr} P(z_0) \sigma \Lambda < \operatorname{tr} P(z_0) \sigma_0.$$

From (9) and (22) we deduce that

$$0 < \sigma_0 \leq q(z_0) Lu(z_0) \leq q(z_0) L\gamma_\sigma^\varepsilon(z_0) \leq \sigma \Lambda < \sigma_0,$$

and this is a contradiction. ■

## 2 The Riemannian case

Let  $f: M^n \rightarrow \bar{M}^{n+1}$  denote an isometric immersion between Riemannian manifolds. Assume that the hypersurface  $f$  is two-sided, that is, there exists a globally defined unit normal vector field  $N$ . Denote by  $A = A_N$  the second fundamental form of  $f$  for the given orientation. Then, the  $k$ -mean curvature  $H_k$  is given by

$$\binom{n}{k} H_k = S_k, \quad 0 \leq k \leq n,$$

where  $S_0 = 1$  and  $S_k$  for  $k \geq 1$  is the  $k$ -symmetric elementary function on the principal curvatures of  $f$ . In particular, when  $k = 1$  then  $H_1 = H$  is the mean curvature of  $f$ . Moreover, for  $k$  even the sign of  $S_k$  (and hence  $H_k$ ) does not depend on the chosen orientation.

The Newton tensors  $P_k: TM \rightarrow TM$ ,  $0 \leq k \leq n$ , arising from  $A$  are defined inductively by  $P_0 = I$  and  $P_k = S_k I - A P_{k-1}$ . Then,

$$\text{Tr} P_k = (n - k) S_k = c_k H_k \quad \text{and} \quad \text{Tr} A P_k = (k + 1) S_{k+1} = c_k H_{k+1} \quad (23)$$

where  $c_k = (n - k) \binom{n}{k} = (k + 1) \binom{n}{k+1}$ .

The second order differential operators  $L_k: \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$  arise from normal variations of  $P_{k+1}$  and are given by

$$L_k = \text{Tr} (P_k \circ \text{hess}).$$

Then, the operator  $L_k$  is semi-elliptic (respectively, elliptic) if and only if  $P_k$  is positive semi-definite (respectively, positive definite).

Let  $B_{\bar{M}}(r)$  denote the geodesic ball with radius  $r$  centered at a reference point  $o \in \bar{M}^{n+1}$ . In the sequel, we assume that the radial sectional curvatures in  $B_{\bar{M}}(r)$  along the geodesics issuing from  $o$  are bounded as  $K_{\bar{M}}^{\text{rad}} \leq b$  for some constant  $b \in \mathbb{R}$ , and that  $0 < r < \min\{\text{inj}_{\bar{M}}(o), \pi/2\sqrt{b}\}$  where  $\text{inj}_{\bar{M}}(o)$  is the injectivity radius at  $o$  and  $\pi/2\sqrt{b}$  is replaced by  $+\infty$  if  $b \leq 0$ .

It is a standard fact that if  $\bar{M}^{n+1}$  has constant sectional curvature  $b$ , then the mean curvature of the geodesic sphere  $S_{\bar{M}}(r) = \partial B_{\bar{M}}(r)$  is

$$C_b(r) = \begin{cases} \sqrt{b} \cot(\sqrt{b} r) & \text{if } b > 0, \\ 1/r & \text{if } b = 0, \\ \sqrt{-b} \coth(\sqrt{-b} r) & \text{if } b < 0. \end{cases} \quad (24)$$

The following classical Hessian comparison result plays an important role in the proof of our results.

**Lemma 4.** *Let  $\bar{M}$  be a Riemannian manifold with a fixed reference point  $o \in \bar{M}$  and let  $\rho(x)$  be the distance function to  $x$ . Let  $x \in \bar{M}$  be inside a geodesic ball  $B_{\bar{M}}(r)$  as above with  $K_{\bar{M}}^{\text{rad}} \leq b$ . Then, for any vector  $X \in T_x \bar{M}$  we have*

$$\text{Hess } \rho(X, X) \geq C_b(\rho(x))(\|X\|^2 - \langle X, \bar{\nabla} \rho(x) \rangle^2)$$

where  $\text{Hess } \rho$  stands for the Hessian of  $\rho$ .

In the following result, it is convenient to think that  $S_{\bar{M}}(r)$  is the smallest possible geodesic sphere centered at  $o$  enclosing the hypersurface.

**Theorem 5.** *Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be a two-sided isometric immersion between complete manifolds where  $M^n$  satisfies condition (2). Assume that  $P_k$  is positive semi-definite for some  $0 \leq k \leq n-1$  and that  $\text{tr} P_k > 0$  on  $M^n$ . If  $f(M) \subset B_{\bar{M}}(r)$  for a geodesic ball  $B_{\bar{M}}(r)$  as above, then*

$$\sup_M \left( \frac{|H_{k+1}|}{H_k} \right) \geq C_b(r). \quad (25)$$

Moreover, if  $P_k$  is positive definite and there exists a point  $p_0 \in M^n$  such that  $f(p_0) \in S_{\bar{M}}(r)$  then equality in (25) implies  $M^n = S_{\bar{M}}(r)$ .

In particular, we have the following consequence.

**Corollary 6.** *Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be as above. Assume that  $P_k$  is positive semi-definite for some  $0 \leq k \leq n-1$ . If  $f(M) \subset B_{\bar{M}}(r)$  for a geodesic ball  $B_{\bar{M}}(r)$  as above, then*

$$\sup_M |H_{k+1}| \geq C_b(r) \inf_M H_k. \quad (26)$$

For the proof of Corollary 6 we first observe that (26) holds trivially if  $\inf_M H_k = 0$ . For  $\inf_M H_k > 0$ , we have that  $P_k \neq 0$  everywhere and the result follows directly from Theorem 5 since (26) is weaker than (25).

*Proof of Theorem 5:* We denote by  $\rho: \bar{M}^{n+1} \rightarrow \mathbb{R}$  the distance function to the reference point  $o$  and set  $u = \rho \circ f$ . Along  $M^n$  we have

$$\bar{\nabla} \rho = \nabla u + \langle \bar{\nabla} \rho, N \rangle N$$

where  $N$  is a unit global normal vector field to  $f$ . An easy computation gives

$$\text{Hess } u(X, Y) = \text{Hess } \rho(X, Y) + \langle \bar{\nabla} \rho, N \rangle \langle AX, Y \rangle$$

where we denoted  $A = A_N$ .

Let  $e_1, \dots, e_n$  be an orthonormal basis of principal directions at a point of  $M^n$ . We obtain using (23) that

$$\begin{aligned} L_k u &= \sum_{i=1}^n \text{Hess } u(e_i, P_k e_i) = \sum_{i=1}^n \text{Hess } \rho(e_i, P_k e_i) + \langle \bar{\nabla} \rho, N \rangle \text{Tr } A P_k \\ &= \sum_{i=1}^n \text{Hess } \rho(e_i, P_k e_i) + c_k H_{k+1} \langle \bar{\nabla} \rho, N \rangle. \end{aligned}$$

By assumption, we have

$$P_k e_i = \mu_i e_i \quad \text{with} \quad \mu_i \geq 0.$$

Using the Hessian comparison theorem, we obtain

$$\begin{aligned} \text{Hess } \rho(e_i, P_k e_i) &= \mu_i \text{Hess } \rho(e_i, e_i) \\ &\geq \mu_i C_b(u) (1 - \langle \nabla u, e_i \rangle^2) \\ &= C_b(u) (\mu_i - \langle \nabla u, e_i \rangle \langle P_k \nabla u, e_i \rangle). \end{aligned}$$

Using (23) we have

$$\begin{aligned} \sum_{i=1}^n \text{Hess } \rho(e_i, P_k e_i) &\geq C_b(u) (\text{Tr } P_k - \langle \nabla u, P_k \nabla u \rangle) \\ &= C_b(u) (c_k H_k - \langle \nabla u, P_k \nabla u \rangle). \end{aligned}$$

Therefore,

$$L_k u \geq C_b(u) (c_k H_k - \langle \nabla u, P_k \nabla u \rangle) + c_k H_{k+1} \langle \bar{\nabla} \rho, N \rangle. \quad (27)$$

Consider the function

$$\phi_b(t) = \begin{cases} 1 - \cos(\sqrt{b}t) & \text{if } b > 0, \\ t^2 & \text{if } b = 0, \\ \coth(\sqrt{-b}t) & \text{if } b < 0. \end{cases}$$

Then  $\phi'_b(t) > 0$  if  $t > 0$  and

$$\phi''_b(t) - C_b(t) \phi'_b(t) = 0. \quad (28)$$

We have using (28) that

$$\begin{aligned} L_k \phi_b(u) &= \phi''_b(u) \langle \nabla u, P_k \nabla u \rangle + \phi'_b(u) L_k u \\ &= \phi'_b(u) (C_b(u) \langle \nabla u, P_k \nabla u \rangle + L_k u). \end{aligned}$$

It follows from (27) that

$$L_k \phi_b(u) \geq c_k \phi'_b(u) (C_b(u) H_k + \langle \bar{\nabla} \rho, N \rangle H_{k+1}).$$

Hence,

$$L_k \phi_b(u) \geq c_k \phi'_b(u) (C_b(u) H_k - |H_{k+1}|).$$

Since  $\sup_M \phi_b(u) \leq \phi_b(r) < +\infty$ , it follows from Theorem 3 that there exists a sequence of points  $\{p_j\}_{j \in \mathbb{N}}$  in  $M^n$  such that

$$\phi_b(u(p_j)) > \sup_M \phi_b(u) - \frac{1}{j} \quad \text{and} \quad \frac{1}{c_k H_k(p_j)} L_k \phi_b(u)(p_j) < \frac{1}{j}.$$

It follows from the first inequality that

$$\lim_{j \rightarrow \infty} u(p_j) = u^* = \sup_M u \tag{29}$$

since  $\sup_M \phi_b(u) = \phi_b(\sup_M u)$ . Therefore,

$$\begin{aligned} \frac{1}{j} &> \frac{1}{c_k H_k(p_j)} L_k \phi_b(u)(p_j) \geq \phi'_b(u(p_j)) \left( C_b(u(p_j)) - \frac{|H_{k+1}|}{H_k}(p_j) \right) \\ &\geq \phi'_b(u(p_j)) \left( C_b(r) - \sup_M \left( \frac{|H_{k+1}|}{H_k} \right) \right) \end{aligned}$$

since  $C_b(u(p_j)) \geq C_b(r)$ . Taking  $j \rightarrow +\infty$  and using (29) we conclude that

$$C_b(r) - \sup_M \left( \frac{|H_{k+1}|}{H_k} \right) \leq 0.$$

For the proof of the second statement, first observe that equality in (25) yields  $L_k \phi_b(u) \geq 0$ . Since  $\phi_b(u) \leq \phi_b(r) < +\infty$ , it follows from the maximum principle for the elliptic operator  $L_k$  that  $\phi_b(u)$  is constant, and hence  $u$  is constant. ■

**Remark 7.** Notice that the conclusion (ii) in Theorem 2 has not been used in the proof of Theorem 5. In this situation, the usual terminology is that we only need a *weak* Omori-Yau maximum principle for trace operators. It turns out that for spacelike hypersurfaces in Lorentzian ambient spaces this is not longer the case.

In the sequel, we replace some assumptions in Theorem 5 by simpler ones and of a geometric nature. This, of course, is the case of Theorem 1 in the Introduction. But first we considered the special case of  $H_2$ . The short proofs given next are mostly taken from [4] and are included for the sake of completeness.

**Corollary 8.** *Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be an isometric immersion into a complete Riemannian manifold. Assume that  $M^n$  is complete with sectional curvature  $K_M \geq K > -\infty$ . If  $H_2 > 0$  and  $f(M) \subset B_{\bar{M}}(r)$  for a geodesic ball  $B_{\bar{M}}(r)$  as above, then*

$$\sup_M \sqrt{H_2} \geq \sup_M \left( \frac{H_2}{H} \right) \geq C_b(r). \quad (30)$$

*If there exists a point  $p_0 \in M^n$  such that  $f(p_0) \in S_{\bar{M}}(r)$  and it holds that  $\text{char}_M(H_2/H) = C_b(r)$ , then  $M^n = S_{\bar{M}}(r)$ .*

*Proof:* In term of the principal curvatures  $\lambda_1, \dots, \lambda_n$  of  $f$  we have that

$$n^2 H^2 = \sum_{j=1}^n \lambda_j^2 + n(n-1)H_2 > \lambda_i^2.$$

In particular, the immersion is two-sided since  $H^2 > 0$ . Moreover, we have that the eigenvalues of  $P_1$  satisfy  $\mu_j = nH - \lambda_j > 0$  for any  $j$  (see Lemma 3.10 in [8]) and therefore  $L_1$  is elliptic. Then, the second inequality and the characterization of equality follows from Theorem 5. For the first inequality, just observe that  $H^2 - H_2 \geq 0$  yields  $H_2/H \leq \sqrt{H_2}$ . ■

**Remark 9.** If the ambient space has constant sectional curvature  $b$ , then the normalized scalar curvature  $s$  of  $M^n$  is related to  $H_2$  by  $s = b + H_2$ . In this case inequality (30) gives

$$\sup_M s \geq b + C_b(r) \inf_M H.$$

*Proof of Theorem 1:* The existence of an elliptic point implies that  $H_{k+1}$  is positive at that point, and hence on  $M^n$ . The well-known Garding inequalities yield, for the appropriate orientation, that

$$H_1 \geq H_2^{1/2} \geq \dots \geq H_k^{1/k} \geq H_{k+1}^{1/(k+1)} > 0. \quad (31)$$

Thus, the immersion is two-sided and  $H_1 > 0$ . Moreover, since  $M^n$  has an elliptic point and  $H_{k+1} \neq 0$  on  $M^n$ , from the proof of [5, Proposition 3.2] we have that the operators  $L_j$  are elliptic for any  $1 \leq j \leq k$ . Then, the second inequality and the characterization of the equality case follows from Theorem 5. For the first inequality observe that  $H_{j+1}/H_j \leq \sqrt[j+1]{H_{j+1}}$  follows from (31). ■

### 3 The Lorentzian case

Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be a spacelike hypersurface isometrically immersed into a spacetime. Since  $\bar{M}^{n+1}$  is time-oriented, there exists a unique globally defined future-directed timelike normal unit vector  $N$ . We refer to  $N$  as the future-directed Gauss map of  $M^n$  and denote by  $A = A_N$  the second fundamental form of the hypersurface.

For spacelike hypersurfaces, the  $k$ -mean curvature  $H_k$  is defined by

$$\binom{n}{k} H_k = (-1)^k S_k, \quad 0 \leq k \leq n,$$

where  $S_0 = 1$  and  $S_k$  for  $k \geq 1$  is the  $k$ -symmetric elementary function on the principal curvatures of  $f$ . The choice of the sign  $(-1)^k$  in the definition is to have the mean curvature vector given by  $\vec{H} = HN$ . Therefore,  $H(p) > 0$  at  $p \in M^n$  if and only if  $\vec{H}(p)$  is future-directed. Clearly, when  $k$  is even the sign of  $H_k$  does not depend on the chosen Gauss map.

For spacelike hypersurfaces, the Newton tensors  $P_k: TM \rightarrow TM$  are defined inductively by  $P_0 = I$  and  $P_k = (-1)^k S_k I + A P_{k-1}$ ,  $1 \leq k \leq n$ . Then,

$$\text{Tr} P_k = c_k H_k \quad \text{and} \quad \text{Tr} A P_k = -c_k H_{k+1}. \quad (32)$$

Let  $o \in \bar{M}^{n+1}$  be a reference point and  $\rho: \bar{M}^{n+1} \rightarrow [0, +\infty]$  the Lorentzian distance from  $o$ . It is well known that the Lorentzian distance function may fail to be continuous and even finite valued. Thus, to guarantee smoothness we need to restrict  $\rho$  to certain special subsets of  $\bar{M}^{n+1}$ . Following [7] (see also [2]) we denote by  $\mathcal{I}^+(o) \subset \bar{M}^{n+1}$  the diffeomorphic image of  $\text{int}(\tilde{\mathcal{I}}^+(o))$  under the exponential map at  $o$ . Here,

$$\tilde{\mathcal{I}}^+(o) = \{tv \in T_o \bar{M} : v \text{ future-directed unit vector and } 0 < t < s_o(v)\}$$

where

$$s_o(v) = \sup\{t \geq 0 : \rho(\gamma_v(t)) = t = L(\gamma_v|_{[0,t]})\}.$$

It turns out that  $\mathcal{I}^+(o)$  is the largest natural open subset of  $\bar{M}^{n+1}$  on which  $\rho$  is smooth and that  $\bar{\nabla}\rho$  is a past-directed timelike (geodesic) unit vector field on  $\mathcal{I}^+(o)$ . We refer to [2], [7] and references therein for further details about the Lorentzian distance function.

For  $b \in \mathbb{R}$ , we consider the function  $\hat{C}_b(t) = C_{-b}(t)$ . We point out that when  $\mathcal{I}^+(o) \neq \emptyset$ , then  $\hat{C}_b(r)$  is the future mean curvature of the level set

$$\Sigma_b(r) = \{x \in \mathcal{I}^+(o) : \rho(x) = r\}$$

in a Lorentzian space form  $\bar{M}_b^{n+1}$  with constant sectional curvature  $b$ . The following Hessian comparison result plays an important role in the proof of our results.

**Lemma 10.** *Assume that  $\mathcal{I}^+(o) \neq \emptyset$  for a reference point  $o \in \bar{M}^{n+1}$ . Let  $x \in \mathcal{I}^+(o)$  and assume that the radial sectional curvatures of  $\bar{M}^{n+1}$  along the radial future geodesic from  $o$  to  $x$  are bounded as  $K_{\bar{M}}^{\text{rad}} \leq b$  (respectively,  $K_{\bar{M}}^{\text{rad}} \geq b$ ) for some constant  $b$ , with  $\rho(x) < \pi/\sqrt{-b}$  if  $b < 0$ . Then, for any spacelike vector  $X \in T_x M$  we have*

$$\text{Hess } \rho(X, X) \geq -\hat{C}_b(\rho(x))(\|X\|^2 + \langle X, \bar{\nabla}\rho(x) \rangle^2) \quad (\text{respectively, } \leq)$$

where  $\text{Hess } \rho$  stands for the Lorentzian Hessian of  $\rho$ .

The proof of Lemma 10 follows easily from the proofs of Lemma 3.1 and Lemma 3.2 in [2] by observing that the assumption  $K_{\bar{M}} \leq b$  (respectively,  $K_{\bar{M}} \geq b$ ) for all timelike planes in  $\bar{M}^{n+1}$  in those results is now needed only for the radial sectional curvatures along the radial future geodesic starting at  $o$ . Observe also that

$$\text{Hess } \rho(X, X) = \text{Hess } \rho(X^*, X^*)$$

where  $X = X^* - \langle X, \bar{\nabla}\rho(x) \rangle \bar{\nabla}\rho(x)$  with  $\langle X^*, \bar{\nabla}\rho(x) \rangle = 0$  and

$$\|X^*\|^2 = \|X\|^2 + \langle X, \bar{\nabla}\rho(x) \rangle^2.$$

For details, see [2, Section 3].

For a given reference point  $o \in \bar{M}^{n+1}$  and  $r > 0$ , let  $B^+(o, r)$  denote the future inner ball of radius  $r$ , namely,

$$B^+(o, r) = \{x \in I^+(o) : \rho(x) < r\},$$

where  $I^+(o)$  is the chronological future of  $o$ , i.e., the set of points  $x \in \bar{M}^{n+1}$  for which there exists a future-directed timelike curve from  $o$  to  $x$ . Now we are ready to state our first result in this section.



**Theorem 11.** *Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be a spacelike hypersurface immersed into a spacetime, where  $M^n$  is complete and satisfies condition (2). Assume that  $P_k$  is positive semi-definite for some  $0 \leq k \leq n-1$  and that  $\text{tr} P_k > 0$  on  $M^n$ . If  $f(M) \subset \mathcal{I}^+(o) \cap B^+(o, r)$  for a reference point  $o \in \bar{M}^{n+1}$  and the radial sectional curvatures along the radial future geodesics issuing from  $o$  are bounded as  $K_M^{\text{rad}} \leq b$  on  $\mathcal{I}^+(o) \cap B^+(o, r)$  for some constant  $b$  (with  $r < \pi/2\sqrt{-b}$  if  $b < 0$ ), then*

$$\inf_M \left( \frac{H_{k+1}}{H_k} \right) \leq \hat{C}_b(u^*), \quad (33)$$

where  $u^* = \sup_M u$ .

*Proof:* We set  $u = \rho \circ f$ . Along  $M^n$ , we have

$$\bar{\nabla} \rho = \nabla u - \langle \bar{\nabla} \rho, N \rangle N = \nabla u - \sqrt{1 + \|\nabla u\|^2} N.$$

Then,

$$\text{Hess } u(X, Y) = \text{Hess } \rho(X, Y) - \sqrt{1 + \|\nabla u\|^2} \langle AX, Y \rangle.$$

A similar computation as in the Riemannian case yields

$$L_k u = \sum_{i=1}^n \text{Hess } \rho(e_i, P_k e_i) + c_k H_{k+1} \sqrt{1 + \|\nabla u\|^2},$$

where  $e_1, \dots, e_n$  is an orthonormal basis of principal directions at a point of  $M^n$ . Using Lemma 10 and reasoning as we did to conclude (27), we obtain

$$L_k u \geq -\hat{C}_b(u)(c_k H_k + \langle \nabla u, P_k \nabla u \rangle) + c_k H_{k+1} \sqrt{1 + \|\nabla u\|^2},$$

where the restriction  $u < \pi/2\sqrt{-b}$  if  $b < 0$  is necessary to have  $\hat{C}_b(u) > 0$ .

Since  $\sup_M u = u^* < +\infty$ , by Theorem 3 there exists a sequence  $\{p_j\}_{j \in \mathbb{N}}$  in  $M^n$  such that

$$u(p_j) > u^* - \frac{1}{j}, \quad \|\nabla u(p_j)\| < \frac{1}{j} \quad \text{and} \quad \frac{1}{c_k H_k(p_j)} L_k u(p_j) < \frac{1}{j}.$$

In particular,  $\lim_{j \rightarrow \infty} u(p_j) = u^*$  and  $\lim_{j \rightarrow \infty} \|\nabla u(p_j)\| = 0$ . Thus,

$$\begin{aligned} \frac{1}{j} &> \frac{1}{c_k H_k(p_j)} L_k u(p_j) \\ &\geq -\hat{C}_b(u(p_j)) \left( 1 + \frac{\langle \nabla u, P_k \nabla u \rangle}{c_k H_k} (p_j) \right) + \frac{H_{k+1}}{H_k} (p_j) \sqrt{1 + \|\nabla u(p_j)\|^2} \\ &\geq -\hat{C}_b(u(p_j)) (1 + \|\nabla u(p_j)\|^2) + \inf_M \left( \frac{H_{k+1}}{H_k} \right) \sqrt{1 + \|\nabla u(p_j)\|^2}, \end{aligned}$$

since, being  $P_k$  positive semi-definite, we have

$$0 \leq \langle X, P_k X \rangle \leq \text{Tr} P_k \|X\|^2 = c_k H_k \|X\|^2$$

for any  $X \in TM$ . Finally, taking  $j \rightarrow +\infty$  we conclude that (33) holds. ■

The following is the second main result in this section.

**Theorem 12.** *Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be a spacelike hypersurface immersed into a spacetime, where  $M^n$  is complete and satisfies condition (2). Assume that  $P_k$  is positive semi-definite for some  $0 \leq k \leq n-1$  and that  $\text{tr} P_k > 0$  on  $M^n$ . If  $f(M) \subset \mathcal{I}^+(o)$  for a reference point  $o \in \bar{M}^{n+1}$  and the radial sectional curvatures along the radial future geodesics issuing from  $o$  are bounded as  $K_M^{\text{rad}} \geq b$  on  $\mathcal{I}^+(o)$  for some constant  $b$  (with  $r < \pi/2\sqrt{-b}$  if  $b < 0$ ), then*

$$\sup_M \left( \frac{H_{k+1}}{H_k} \right) \geq \hat{C}_b(u_*), \quad (34)$$

where  $u_* = \inf_M u$ . In particular, if  $u_* = 0$  then  $\sup_M (H_{k+1}/H_k) = +\infty$ .

*Proof:* We proceed as in the proof of Theorem 11, by observing that in this case Lemma 10 yields

$$\begin{aligned} L_k u &\leq -\hat{C}_b(u)(c_k H_k + \langle \nabla u, P_k \nabla u \rangle) + c_k H_{k+1} \sqrt{1 + \|\nabla u\|^2} \\ &\leq -c_k H_k \hat{C}_b(u) + c_k H_{k+1} \sqrt{1 + \|\nabla u\|^2}. \end{aligned}$$

Since  $\inf_M u = u_* \geq 0$ , by Theorem 3 there is a sequence  $\{p_j\}_{j \in \mathbb{N}} \subset M^n$  such that

$$u(p_j) < u_* + \frac{1}{j}, \quad \|\nabla u(p_j)\| < \frac{1}{j} \quad \text{and} \quad \frac{1}{c_k H_k(p_j)} L_k u(p_j) > -\frac{1}{j}.$$

In particular,  $\lim_{j \rightarrow \infty} u(p_j) = u_*$  and  $\lim_{j \rightarrow \infty} \|\nabla u(p_j)\| = 0$ . Thus

$$\begin{aligned} -\frac{1}{j} &< \frac{1}{c_k H_k(p_j)} L_k u(p_j) \\ &\leq -\hat{C}_b(u(p_j)) + \frac{H_{k+1}}{H_k}(p_j) \sqrt{1 + \|\nabla u(p_j)\|^2} \\ &\leq -\hat{C}_b(u(p_j)) + \sup_M \left( \frac{H_{k+1}}{H_k} \right) \sqrt{1 + \|\nabla u(p_j)\|^2}, \end{aligned}$$

and we conclude taking  $j \rightarrow +\infty$  that (34) holds. The last assertion follows from (34) and the fact that  $\lim_{t \rightarrow 0^+} \hat{C}_b(t) = +\infty$ . ■

As a direct application of Theorem 12 we get the following result.

**Corollary 13.** *Under the assumptions of Theorem 12, assume also that  $H_{k+1}/H_k$  is bounded from above on  $M^n$ . Then, there exists  $\delta > 0$  such that  $f(M) \subset O^+(o, \delta)$ , where  $O^+(o, \delta)$  denotes the future outer ball of radius  $\delta$  given by*

$$O^+(o, \delta) = \{x \in I^+(o) : \rho(x) > \delta\}.$$

*Proof:* Simply observe that  $\sup_M (H_{k+1}/H_k) < +\infty$  implies  $u_* > 0$ . ■

If the ambient spacetime is a Lorentzian space form, from Theorem 11 and Theorem 12 we obtain the following consequence that extends Theorem 4.5 in [2] to the case of higher order mean curvatures.

**Corollary 14.** *Let  $f: M^n \rightarrow \bar{M}^{n+1}$  be a spacelike hypersurface immersed into a Lorentzian spacetime of constant sectional curvature  $b$ , where  $M^n$  is complete and satisfies condition (2). Assume that  $P_k$  is positive semi-definite for some  $0 \leq k \leq n-1$  and that  $\text{tr} P_k > 0$  on  $M^n$ . If  $f(M) \subset \mathcal{I}^+(o) \cap B^+(o, r)$  for a reference point  $o \in \bar{M}$  (with  $r < \pi/2\sqrt{-b}$  if  $b < 0$ ), then*

$$\inf_M \left( \frac{H_{k+1}}{H_k} \right) \leq \widehat{C}_b(u^*) \leq \widehat{C}_b(u_*) \leq \sup_M \left( \frac{H_{k+1}}{H_k} \right). \quad (35)$$

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